Supports of multi-microlocalizations with growth conditions

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Abstract

We define the notions of strong regularity of subanalytic sheaves, and prove an estimation of supports of multi-microlocalizations for strongly regular subanalytic sheaves. We apply the results to the subanalytic sheaves of Whitney and temperate holomorphic solutions of regular $\mathcal{D}_X$-modules.

1 Introduction

Sato, Kawai and Kashiwara introduced a new method of analysis, algebraic analysis and microlocal analysis([3]). This theory concerns the microlocal properties of a function.

Kashiwara and Schapira developed the microlocal theory of sheaves([4]). They introduced the notion of the microsupport $\mathrm{SS}(F)$ for a sheaf F on a derived category of sheaves. This is a formalization of the singularity of a function using the theory of sheaves, and it is the collection of directions in which the cohomology of the sheaf F varies. This work provided a powerful new framework for studying singularities of solutions to differential equations.

On the other hand, important analytical objects such as asymptotically developable functions and tempered distributions do not form classical sheaves, therefore conventional methods cannot be applied. New concepts such as ind-sheaves and subanalytic sheaves have been introduced and studied to handle these objects([15, 8]). In [1], Honda, Prelli and Yamazaki constructed the theory of multi-microlocalizations of subanalytic sheaves along a certain family of submanifolds. Their framework allows us to handle multi-microlocal objects with growth conditions such as a sheaf of Majima's asymptotically developable functions functorially(see [10, 2]). In particular they obtained estimate of microsupport of multi-microlocalizations by that of a sheaf F. However their estimation is limited to usual sheaves. For this limitation, the multi-microlocalization under growth conditions is a difficult object to understand microlocally.

The notion of microsupport for ind-sheaves was defined in [6]. And its functorial properties was studied in [5]. In [9], Prelli obtained an estimate of supports of microlocalizations for subanalytic sheaves by means of the microsupport in terms of Kashiwara and Schapira([6]). After some study we found that it is necessary to strengthen the microlocal conditions to find the estimate of supports of multi-microlocalizations for subanalytic sheaves. In this article, we introduce the possible solutions to find an estimate of supports of multi-microlocalizations for subanalytic sheaves.

One of the possible solutions is introducing the notion of stronger regularity than that of Kashiwara and Schapira's notion introduced in [6]. In [5], Martins proved that the regularity of ind-sheaves has a nice relation with regular $\mathcal{D}$-modules along an involutive submanifold in the sense of Kashiwara and Oshima ([14]). We found that strengthening the regularity is still valid to have the good relation with the regular $\mathcal{D}$-modules. Applying this fact, we deduce some properties of multi-microlocal analytic objects including a Majima's strong asymptotically developable solution of regular $\mathcal{D}$-modules.

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1.1 Multi-microlocal analysis

In this section we recall some fundamental preliminaries. We basically follow the notations of [4] and [1].

For a sheaf F on a manifold X, one may associate a conic object $\nu_M(F)$ on the normal bundle T_MX along a closed submanifold M. We call this functor $\nu_M(F)$ a specialization along M. One may send a conic object on T_MX to the conormal bundle T^*_MX. This functor is called Fourier-Sato transformation and denoted by $(\bullet)^\wedge$. Combining these two functors we have a microlocalization functor $\mu_M := (\nu_M(\bullet))^\wedge$ along M. Moreover one can obtain multi-microlocalization $\mu_\chi := (\nu_\chi(\bullet))^\wedge$ along a family of closed manifolds χ with growth condtions(see [1] for details). By microlocalizations we may have the following distinguished triangle:

Theorem 1.1 (Sato triangle)**.** Let $F \in D^b(k_X)$.

$$F|_M \otimes \mathscr{or}_{M/X}[-n] \to R\Gamma_M(F)|_M \to R\dot{\pi}_*\mu_M(F) \to +1$$

is a distinguished triangle here $\dot{\pi} : T^*_MX \setminus M \to M$.

Let X be a complex neighborhood of a manifold M.

Theorem 1.2 ([3])**.** We have $H^j(\mu_M(\mathscr{O}_X)) = 0, j \neq n$.

$C_M := H^n(\mu_M(\mathscr{O}_X)) \otimes \mathscr{or}_{M/X}$ is a sheaf of microfunctions on M and $\mathcal{B}_M := H^n(R\Gamma_M(\mathscr{O}_X)|_M) \otimes \mathscr{or}_{M/X}$ is a sheaf of Sato hyperfunctions on M.

By this theorem we obtain the following well-known Sato exact sequence:

$$0 \to \mathcal{A}_M \to \mathcal{B}_M \xrightarrow{sp} \dot{\pi}_*C_M \to 0,$$

where $\mathcal{A}_M$ is a sheaf of analytic functions on M.

Remark 1.3. Thus roughly speaking, $\mu_M(\mathscr{O}_X)$ measures the "non-analyticity" of Sato hyperfunctions. As the construction is functorial, this methodology is valid to analyze many analytical objects which form sheaves. However many of the classical analytic objects do not form a sheaf. Moreover the classical microlocal analysis is only valid to analyze the singularity along only one submanifold. To address this limitations, in [6], [9], [1], they obtained a generalized framework of the microlocal theory.

Theorem 1.4. Let $F \in D^b(k_{X_{sa}})$ be an object of the derived category of subanalytic sheaves. We have the following distinguished triangle:

$$\nu_\chi(F) \to \tau^{-1}R\Gamma_M(F) \otimes \mathscr{or}_{M/X}[n] \to Rp^+_{1*}(p^+_2)^{-1}\mu_\chi(F) \otimes \mathscr{or}_{M/X}[n] \xrightarrow{+1},$$

here for p^+, p^+ and τ we do not explain in details(see [1, Theorem 4.2]).

Example 1.5 (Majima's strongly asymptotically developable functions)**.** By applying the multi-specialization functor ν_χ along the family of normal crossing submanifolds χ to the subanalytic sheaf of Whitney holomorphic functions $\mathscr{O}^w$ we obtain a sheaf of Majima's strongly asymptotically developable functions which is defined in [10](see [2] for details).

We can define the functor $\mu hom_{\hat{\chi}}$, a variant of Sato's microlocalization functor(see [13]). We can obtain Sato triangle for the functor $\mu hom_{\hat{\chi}}$ (see [12]):

Theorem 1.6 (Sato triangle in multi-microlocalizations)**.** Let $\hat{\chi}$ be a family of submanifolds with certain condition. For $G_1, G_2, \ldots, G_\ell \in D^b_{\mathbb{R}\text{-}c}(k_X)$ and $F \in D^b(k_{X_{sa}})$, we have the following distinguished triangle:

$$\bigotimes_{i=1}^{\ell} D'G_i \otimes F \to \mathrm{R}\mathscr{H}om(\bigotimes_{i=1}^{\ell} G_i, F) \to R\dot{\pi}_*\mu\mathrm{hom}^{sa}_{\hat{\chi}}(G_1, \ldots, G_\ell; F) \xrightarrow{+1} .$$

2 Main Results: an estimation of supports of microlocalizations with growth conditions

Novelty: All the results in this section are newly obtained by the author.

All families of submanifolds are assumed to be normal crossing type.

2.1 normal crossing type $\chi = \{M_1, M_2\}$

Proposition 2.1. *Assume $F \in D^b(k_X)$.*

$$C_{\chi^*}(\mathrm{SS}(F)) \cap S^*_\chi = \overline{t_1 t_2 \, \mathrm{SS}(F)} \cap T^*_{M_1}X \times_M T^*_{M_2}X$$

In particular, if $\mathrm{SS}(F)$ *is t_1, t_2-conic we have:*

$$C_{\chi^*}(\mathrm{SS}(F)) \cap S^*_\chi = \mathrm{SS}(F) \cap T^*_{M_1}X \times_M T^*_{M_2}X.$$

Proof. $(x_1, x_2; \xi_1, \xi_2) \in C_{\chi^*}(\mathrm{SS}(F))$ then there exists sequence $(x_{1,n}, x_{2,n}; \xi_{1,n}, \xi_{2,n}) \in \mathrm{SS}(F)$ and $(t_{1,n}, t_{2,n}) \subset (\mathbb{R}^+)^2$ such that

$$(t_{1,n}x_{1,n}, t_{2,n}x_{2,n}; t_{2,n}\xi_{1,n}, t_{1,n}\xi_{2,n}) \to (x_1, x_2; \xi_1, \xi_2) \tag{1}$$

and $t_{1,n}, t_{2,n} \to \infty$. This gives the result. □

Definition 2.2 (strongly regular subanalytic sheaves)**.** We say $F \in D^b(k_{X_{sa}})$ is strongly regular along V if there exists a small filtrant system $\{F_i\}$ in $C^{[a,b]}(\mathrm{Mod}^c_{R-c}(k_X))$ with $\mathrm{SS}(F_i) \subset V$ such that F is quasi-isomorphic to $\varinjlim_i \rho_* F_i$ locally on X.

Theorem 2.3. *Let V be a subset in T^*X. Assume $F \in D^b(k_{X_{sa}})$ is strongly regular along V. Then*

$$\mathrm{supp}(\rho^{-1}\mu^{sa}_\chi F) \subset \overline{t_1 t_2 V}.$$

Proof. Let $p = (x; \xi) \notin \overline{t_1 t_2 V}$. By the strong regularity along V, there exist a small filtrant system $\{F_i\}$ in $C^{[a,b]}(\mathrm{Mod}^c_{\mathbb{R}-c}(k_X))$ with $\mathrm{SS}(F_i) \subset V$ such that F is quasi-isomorphic to $\varinjlim_i \rho_* F_i$ on a neighborhood W of x. And also we find a t_1, t_2-conic open neighborhood U of p such that $\overline{t_1 t_2 V} \cap U = \emptyset$ and $U \subset \pi^{-1}(W)$. We have:

$$H^k \mu^{sa}_\chi(F_W) \simeq \varinjlim_i H^k \rho_* \mu_\chi F_{iW}.$$

Then $\mu^{sa}_\chi(F)|_U = 0$ since $\mathrm{supp}(\mu_\chi F_i) \subset \overline{t_1 t_2 \, \mathrm{SS}(F_i)} \subset \overline{t_1 t_2 V}$. □

Proposition 2.4. *Let $G_1, G_2 \in D^b_{\mathbb{R}-c}(k_X)$, $F \in D^b(k_{X_{sa}})$. Assume F is strongly regular along V. Let $\hat{\chi}$ be of a normal crossing type.*

$$\mathrm{supp}\, \mu hom_{\hat{\chi}}(G_1, G_2; F) \subset \overline{t_1 t_2 (\mathrm{SS}(G_1) \times \mathrm{SS}(G_2) \cap i_{\Delta\pi}{}^t i'^{-1}_\Delta V)}.$$

Proof. $\mathrm{R}\mathscr{H}om(p_2^{-1}(G_1 \boxtimes G_2), p_1^! R i_{\Delta *} F))$ is strongly regular along $\mathrm{SS}(G_1) \times \mathrm{SS}(G_2) \cap i_{\Delta\pi}{}^t i'^{-1}_\Delta V$. □

Proposition 2.5. *Let $G_1, G_2 \in D^b_{\mathbb{R}-c}(k_X)$, $F \in D^b(k_{X_{sa}})$. Assume F is strongly regular along V. Let $\hat{\chi}$ be of a normal crossing type. Suppose that* $\mathrm{SS}(G_1) \times \mathrm{SS}(G_2) \cap i_{\Delta\pi}{}^t i'^{-1}_\Delta V \subset T^*_X X \times T^*_X X$. *Then*

$$D'G_1 \otimes D'G_2 \otimes \rho^{-1} F \xrightarrow{\sim} \rho^{-1}\, \mathrm{R}\mathscr{H}om(G_1 \otimes G_2, F).$$

Proof. Use the distinguished triangle in Theorem 1.6. □

2.2 Solution of $\mathcal{D}$-modules in complex domains with growth conditions.

Let $\chi := \{Y_1, Y_2\}$ and assume each Y_i and $Y := \bigcap Y_i$ are complex submanifolds of X. As usual, let $\mathcal{D}_X$ be the sheaf of holomorphic differential operators on X. Assume that Y is non-characteristic for an involutive subbundle V in T^*X and $\mathcal{M}$ is a regular coherent $\mathcal{D}_X$-module along V.

Theorem 2.6. *If $\mathcal{M}$ is regular along an involutive vector subbundle V of T^*X, then* $\mathrm{Sol}^w(\mathcal{M}) := \mathrm{R}\mathcal{H}om(\rho_!\mathcal{M}, \mathcal{O}_X^w)$ *and* $\mathrm{Sol}^t(\mathcal{M}) := \mathrm{R}\mathcal{H}om(\rho_!\mathcal{M}, \mathcal{O}_X^t)$ *are strongly regular along V.*

Proof. We may assume $X = Z \times Y$, for some complex manifolds Z and Y, that f is the projection $X \to Y$ and that $V = X \times_Y T^*Y$. By [11, Lemma 3.6] we may also assume $\mathcal{M} = \mathcal{D}_{X\to Y}$. By [9, Corollary A.4.8], we have

$$f^{-1}\mathcal{O}_Y^w \simeq \mathrm{R}\mathcal{H}om(\rho_!\mathcal{D}_{X\to Y}, \mathcal{O}_X^w).$$

There exists a small filtrant system $\{F_i\} \in C^{[a,b]}(\mathrm{Mod}^c_{R-c}(k_X))$ so that $F := \mathcal{O}_Y^w$ is quasi isomorphic to $\varinjlim_i \rho_* F_i$. Then $f^{-1}F$ is quasi isomorphic to $\varinjlim_i \rho_* f^{-1}F_i$ locally on X and one has

$$\mathrm{SS}(f^{-1}F_i) \subset X \times_Y T^*Y, \text{ for all } i \in I.$$

It follows that $f^{-1}F$ is strongly regular along V.

For the case of $\mathrm{Sol}^t(\mathcal{M})$, the proof is similar([5, Proposition 3.8]). We may use [9, Corollary A.3.7] in this case. □

Corollary 2.7. *Let $F \in D^b(k_{X_{sa}})$. Assume F is strongly regular along V in T^*X. If $\dot{T}^*_{M_1}X \times_X \dot{T}^*_{M_2}X \cap \overline{t_1t_2V} = \emptyset$, then*

$$\nu_\chi(F) \xrightarrow{\sim} \tau^{-1}\rho^{-1}R\Gamma_M(F) \otimes \omega_{M/X}^{\otimes -1}.$$

Proof. By [1, Theorem 4.2] and Theorem 2.3, the result follows. □

Theorem 2.8. *Assume that Y is non-characteristic for an involutive subbundle V in T^*X and $\mathcal{M}$ is regular coherent $\mathcal{D}_X$-module along V. Then*

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{O}_X^\lambda)) \simeq \tau^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{O}_Y) \tag{2}$$

$$\simeq \tau^{-1}f^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \tag{3}$$

where $\lambda = w$ or t.

Proof. By assumption, Y is non-characteristic for $\mathcal{M}$. By the non-characteristic condition and Cauchy-Kowaleskaya-Kashiwara theorem,

$$\tau^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \rho^{-1}R\Gamma_Y(\mathcal{O}_X^\lambda)) \otimes \omega_{Y/X}^{\otimes -1} \simeq \tau^{-1}f^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \tag{4}$$

and

$$\tau^{-1}f^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq \tau^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{O}_Y). \tag{5}$$

Here we used [9, Proposition 5.3.8], [7, Theorem 3.1.1] and [7, Corollary 2.2.3].

Since $\mathrm{Sol}^\lambda(\mathcal{M})$ is strongly regular along V by Theorem 2.6, by Corollary 2.7 it is enough to check $\dot{T}^*_{M_1}X \times_X \dot{T}^*_{M_2}X \cap \overline{t_1t_2V} = \emptyset$. However almost the same fact was already checked in the proof of [1, Theorem 4.4]. □

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